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On the inclined non-inertial sinkage of a flat plate

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An approximate analytical solution is derived for the inclined non-inertial sinkage of a flat plate onto a smooth surface, for the case of small angles. The result is a modification to the existing solutions for parallel sinkage, in the form of a polynomial in the dimensionless angle of inclination. Even the smallest angle of inclination of plate to surface produces a large increase in the sinkage rate otherwise obtained. The results are confirmed experimentally.

Nomenclature

h_0 = initial film thickness, in.
$h(t)$ = instantaneous film thickness, in.
$h(x,t)$ = instantaneous local film thickness, in.
H_0 = initial minimum film thickness, in.
$H(t)$ = instantaneous minimum film thickness, in.
$h_0(x)$ = initial local film thickness, in.
L = characteristic dimension of flat plate, or length of side of square plate, in.
dh/dt = sinkage rate, in./sec
μ = absolute viscosity, lb. sec/in.^{2}
W = applied sinkage load, lb.
α = initial angle of inclination, rad
$\alpha L/H_0$ = dimensionless initial angle of inclination
β = final angle of inclination, rad
$\theta(t)$ = instantaneous angle of inclination, rad
$\theta L/H$ = dimensionless instantaneous angle of inclination
p = pressure, lb./in.^{2}

1. Introduction

Theoretical solutions exist for the parallel sinkage (figure 1(*a*)) of round (Archibald 1956; Mitchell 1950), elliptical (Saal 1936) and rectangular (Hays 1962) flat plates onto smooth surfaces in the absence of inertia effects. In all cases, the time of sinkage and sinkage rate are given by

$$t = \frac{\mu L^4}{2KW}\left[\frac{1}{h^2}-\frac{1}{h_0^2}\right], \tag{1}$$

$$dh/dt = -KWh^3/\mu L^4, \tag{2}$$

where the constant K depends on the shape of the plate.

The sinkage rate may be varied by changing any of the five parameters on the right side of equation (2). If these are held constant, however, only two possibilities exist for changing dh/dt: (i) inclining the plate to the surface during sinkage, or (ii) replacing the smooth surface by a rough surface. Intuitively, both of these methods, either separately or in unison, will increase the sinkage rate and they will decrease the film thickness otherwise attained in a specified time. The

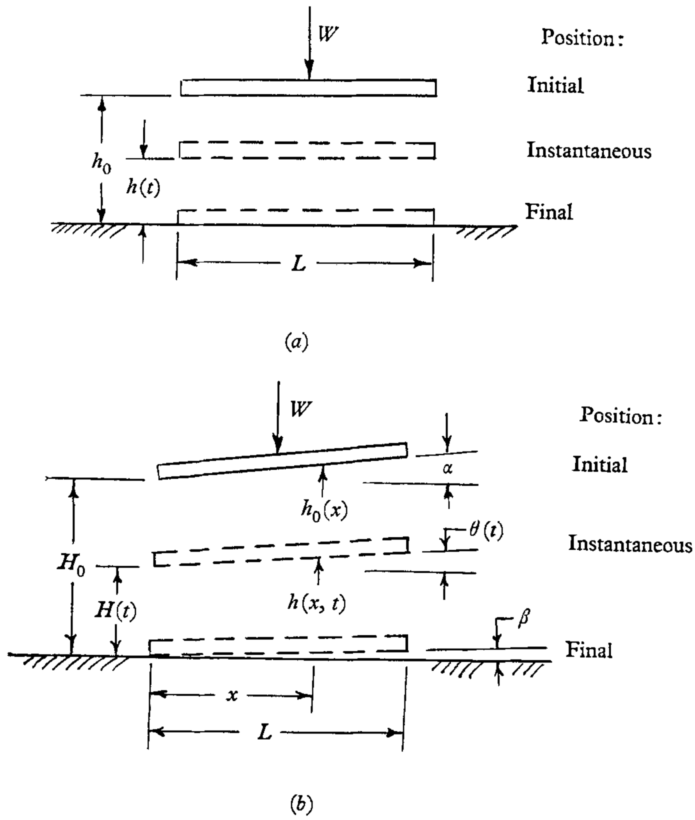

FIGURE 1. (*a*) Parallel sinkage. (*b*) Inclined sinkage.

method of inclining the plate provides, as it were, an angle of attack, and the substitution of a rough surface permits an effective drainage of the film. The problem of surface roughness has been dealt with elsewhere by the author (Moore 1964). This paper will be confined solely to the effect on sinkage rate of inclining the plate during the non-inertial sinkage.

2. Inclined sinkage

In the general case of inclined sinkage, the angle $\theta(t)$ between plate and surface varies with time. Consider the particular case for which $\theta(t)$ decreases progressively and linearly with squeeze film thickness as sinkage occurs, having the values α and β at the start and end of sinkage respectively as shown in figure 1 (*b*). Then

$$H(t)/H_0 = [\theta(t)-\beta]/(\alpha-\beta). \tag{3}$$

Two cases will be considered depending on whether the final angle of inclination β is zero or non-zero. A square plate will be selected for which the value of K in equation (2) is 2·37 (Hays 1962). Since the inclined plate is guided downwards as shown later in figure 5(*b*), it is not possible to have sideslip relative to the squeeze film and so this is not considered in the theory.

3. Theoretical investigation

The Reynolds equation for squeeze films takes the form

$$\frac{\partial}{\partial x}\left[h^3\frac{\partial p}{\partial x}\right]+\frac{\partial}{\partial z}\left[h^3\frac{\partial p}{\partial z}\right] = 12\mu\frac{\partial h}{\partial t} \tag{4}$$

for rectangular plates with a liquid of constant viscosity. The direction x is taken in the plane of the angle of inclination of the plate and parallel to the surface, and z is taken perpendicular to x and along the leading edge of the plate. The origin of co-ordinates coincides with a corner of the plate. From figure 1(*b*),

$$h(x,t) = H(t)+\theta(t)\,x. \tag{5}$$

For the case of a round plate in *parallel* sinkage, the pressure distribution under the sinking plate is a paraboloid of revolution (being constant in time since inertia effects are neglected). For a square or rectangular plate in *parallel* sinkage, however, the assumption of a pressure distribution which is a parabola for any section perpendicular to the x or z direction is only an approximation, since corner effects are thereby neglected. Such a distribution is given by

$$p = \frac{36W}{L^6}\,xz\,(L-x)\,(L-z), \tag{6}$$

for a square plate of side length L. Equation (6) satisfies the condition that the integration of the pressure over the plate area is equal to the load (a valid condition in the absence of inertia) and, of course, the boundary conditions. For parallel sinkage, h is a function of time only, and the substitution of p from equation (6) into equation (4) gives

$$\frac{dh}{dt} = \frac{1}{L^2}\int_0^L\int_0^L\frac{\partial h}{\partial t}\,dx\,dz = -2\frac{Wh^3}{\mu L^4}. \tag{7}$$

If equation (6) were valid, the constant in equation (7) would have been 2·37 (Hays 1962) instead of 2, and thus an error of 15 % is introduced by neglecting corner effects in the assumed pressure distribution.

For *inclined* sinkage of a square plate, there are added complications. Because of the inclination of the plate during sinkage, the pressure distribution in the liquid film becomes asymmetrical in the plane of the inclination. The point of maximum pressure moves a short distance from the geometric plate centre towards the leading edge. With the assumption that this distance is equal to the distance between the centres of gravity and pressure, it is found to be numerically about $1\frac{1}{2}$ % of L for the experimental conditions. The effect of pressure asymmetry will therefore be neglected and equation (6) for a square plate will be assumed to hold.

Since for inclined sinkage h is a function of x and t, equation (4) must be written in the form

$$\frac{1}{L}\int_0^L \left\{\frac{\partial}{\partial x}\left[h^3\frac{\partial p}{\partial x}\right]+\frac{\partial}{\partial z}\left[h^3\frac{\partial p}{\partial z}\right]\right\}dz = 12\mu\frac{\partial h}{\partial t}, \tag{4'}$$

in order that both sides of equation (4') are functions of x and t only. Substituting equations (5) and (6) in equation (4'), and remembering that for inclined sinkage

$$\frac{dh}{dt} = \frac{1}{L}\int_0^L \frac{\partial h}{\partial t}\,dx, \tag{8}$$

it is found that

$$\frac{dh}{dt} = -2\frac{WH^3}{\mu L^4}\left[1+1\cdot 5\left(\frac{\theta L}{H}\right)+1\cdot 2\left(\frac{\theta L}{H}\right)^2+0\cdot 1\left(\frac{\theta L}{H}\right)^3\right]. \tag{9}$$

It is observed that if $\theta = 0$, equation (9) is identical with equation (7) assuming that $H(t)$ is replaced by $h(t)$. If the constant 2 in equation (9) is replaced by 2·37 to eliminate corner effects,

$$\frac{dh}{dt} = -2\cdot 37\frac{WH^3}{\mu L^4}\left[1+1\cdot 5\left(\frac{\theta L}{H}\right)+1\cdot 2\left(\frac{\theta L}{H}\right)^2+0\cdot 1\left(\frac{\theta L}{H}\right)^3\right]. \tag{9'}$$

Then it is apparent that the accuracy of equation (9') is of the same order of magnitude as that obtainable with the experimental apparatus (i.e. within a few %).

Case (a), $\beta = 0$

Equation (3) for this case shows that $(\theta L/H)$ in equation (9') may be replaced by $(\alpha L/H_0)$ which is a constant. Taking $(\alpha L/H_0)$ equal to unity, the value of the polynomial in equation (9') is 3·8. With $L = 9$ in. and $H_0 = \frac{1}{4}$ in. (these being dimensions compatible with the neglect of inertia for very viscous liquids), then $\alpha \doteq 1\cdot 6°$. Thus if $H(t)$ in inclined sinkage is assumed to be interchangeable with $h(t)$ in parallel sinkage, a rapid increase in sinkage rate is obtained by inclining the plate only a few degrees.

Figure 2 shows characteristic dh/dt vs h^3 curves for parallel and inclined sinkage of a square plate onto a smooth surface assuming that $H(t)$ and $h(t)$ are interchangeable. It can also be shown that if $h(t)$ in parallel sinkage is assumed to correspond with the mean height $[H(t)+\frac{1}{2}\theta(t)\,L]$ in inclined sinkage, the inclined case still produces a 13 % increase in sinkage rate over that obtained in parallel sinkage. Figure 3 shows the theoretical effect of changes in $(\alpha L/H_0)$ on dh/dt for values of H equal to 0·1 and 0·08 in.

Case (b), $\beta \neq 0$

For $\beta \neq 0$ and considered positive in the same sense as α, then from equation (3)

$$(\theta L/H) = 1+1/16H, \tag{10'}$$

if $L = 9$ in., $H_0 = \frac{1}{4}$ in., $\alpha = 5/144$ rad and $\beta = 1/44$ rad. The dimensionless angle is now no longer constant but varies with H. If $(\theta L/H)$ is calculated at each H and substituted directly in equation (9'), the sinkage rate can be computed. It is observed that the polynomial now has a different value for each H and the characteristic dh/dt vs h^3 or H^3 curve is no longer a straight line as shown in figure 4.

For β negative and equal to 1/144 rad, and $\alpha = 3/144$ rad, it is found that

$$(\theta L/H) = 1 - 1/16H. \tag{10''}$$

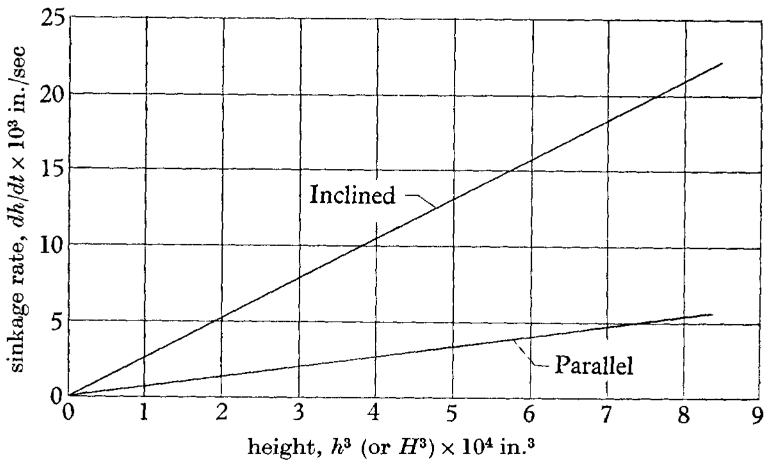

FIGURE 2. Characteristic curves for parallel and inclined sinkage. Square plate in SAE 40 lubricating oil. $\alpha L/H_0 = 1$, $T = 77°$ F, $W = 1$ lb., $L = 9$ in., $H_0 = \frac{1}{4}$ in.

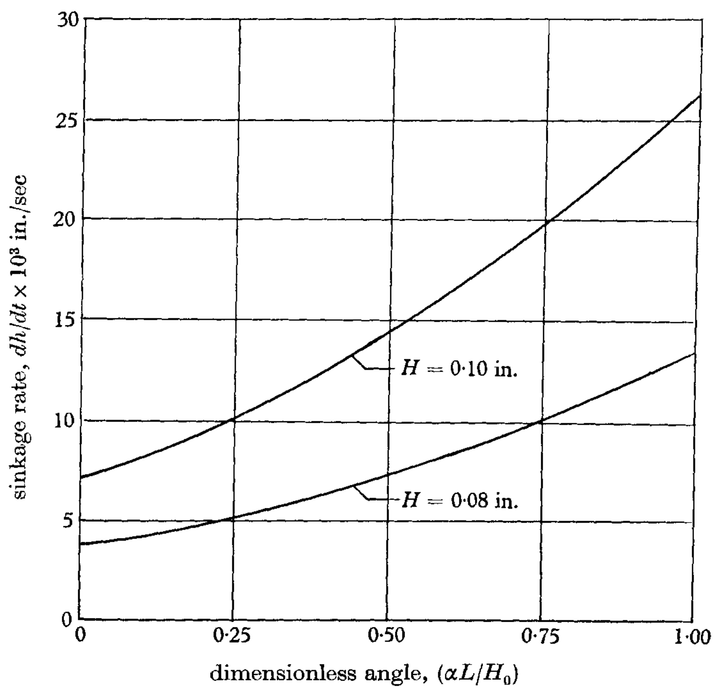

FIGURE 3. Theoretical effect on sinkage rate of changes in angle of inclination. Square plate in SAE 40 lubricating oil. $T = 77°$ F, $W = 1$ lb., $L = 9$ in., $H_0 = \frac{1}{4}$ in.

It is noted in figure 4 that the characteristic dh/dt vs h^3 curves for β positive and negative lie respectively above and below the straight line plot for $\beta = 0$. The value of $(\alpha - \beta)$ is kept constant for all three curves, being equal to 1/36 rad.

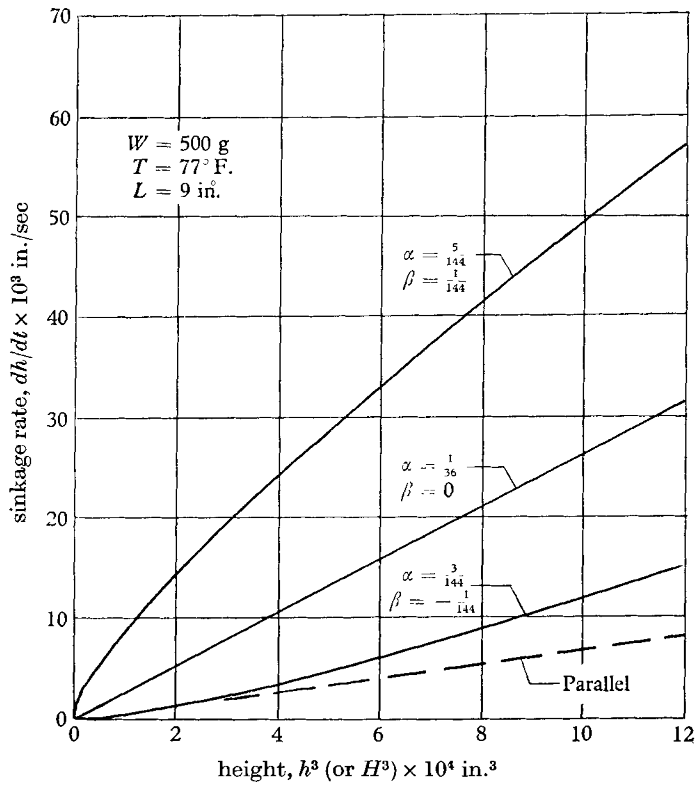

FIGURE 4. Effect of positive, negative, and zero β-angle on sinkage rate. Square plate in SAE 40 lubricating oil. $W = 500$ g, $T = 77$ °F, $L = 9$ in., $H_0 = \frac{1}{4}$ in.

4. Experimental investigation

Experimental confirmation of the reasoning in case (*a*) above for zero β-value is obtained using a sinkage apparatus. This consists essentially of a beam pivoted on roller bearings at its centre and carrying a flat square plate at one end and a system of balance weights at the other end, as indicated in figure 5 (*a*) or (*b*). The linkage arrangement provided to ensure that the sinkage is parallel is shown in figure 5 (*a*). It can be seen that by removing the top linkage member from the position *BA* to the position *BE*, the sinkage becomes inclined as shown in figure 5 (*b*).

To minimize inertia effects, the plate is allowed to sink in a bath of viscous SAE 40 oil. The height of the plate at any instant above the surface is recorded by a mechanical amplification linkage of the rack and pinion type and transmitted to a pointer which moves over a dial reading to thousandths of an inch. Time is recorded by a clock reading to hundredths of a second. Both time and height are recorded simultaneously by a fixed camera with an automatic motor-driven film advance mechanism. When the plate has been balanced in the oil by

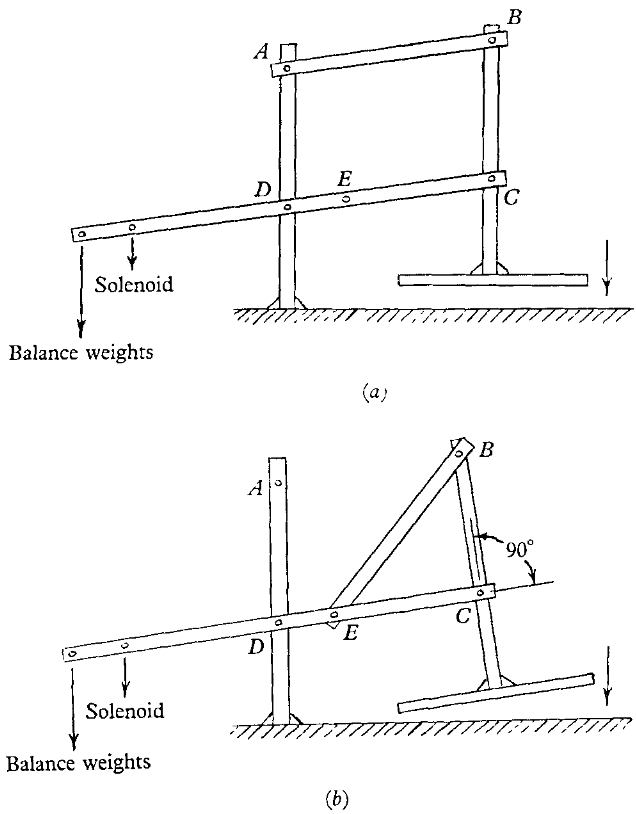

FIGURE 5. (*a*) Linkage for parallel sinkage. (*b*) Linkage for inclined sinkage. *Note.* Roller bearings at *A*, *B*, *C* and *D*.

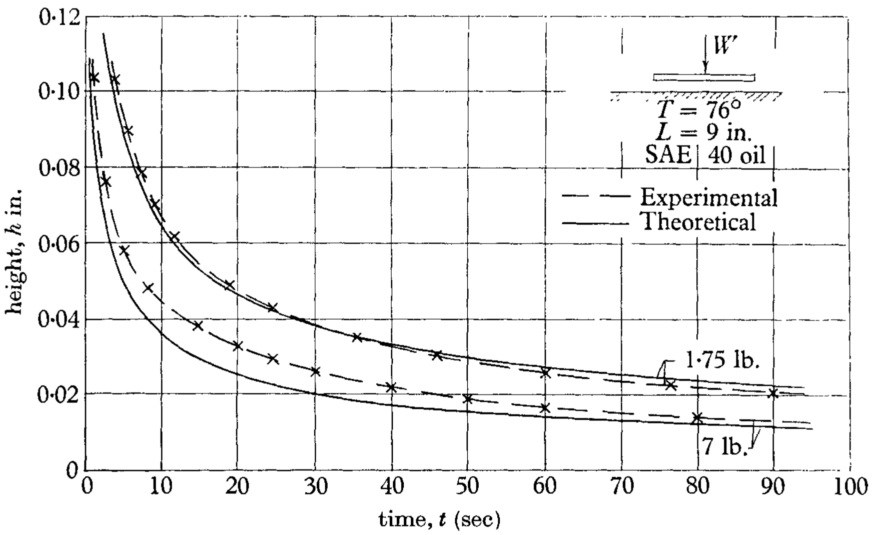

FIGURE 6. Comparison of theoretical and experimental h vs t curves for parallel sinkage.

a suitable selection of weights, a solenoid mechanism is activated which holds the plate prior to sinkage, while the load to be applied to the plate is removed from the weighted end. A throwover switch then initiates sinkage by simultaneously releasing the solenoid and causing the clock to start.

Figure 6 shows a comparison of experimental h vs t curves for parallel sinkage (as obtained with the sinkage model) with the theory of Hays (1962), and figure 7 compares the h vs t curves obtained experimentally with the theory summarized in equation (9′). It is observed at once from these figures that: (i) the sinkage height h decreases very rapidly at first and then more slowly; (ii) at the higher loads, the h *vs* t curves lie consistently below those for the lighter loads; (iii) the experimental curves lie consistently above the corresponding theoretical curves, except at large time values; (iv) the discrepancy between experiment and theory is greatest near the point of maximum curvature of the curves; (v) the discrepancy between experiment and theory increases with increasing load for parallel sinkage, but the reverse is true for inclined sinkage. It will be now shown that these features are entirely systematic and predictable.

5. Discussion

At large height values, the experimental curves are displaced to the right of the corresponding theoretical plots. This is due to the fact that a solenoid releases the plate under load to initiate sinkage, so that the slope of the experimental h vs t curve at the maximum release height H_0 ($= \frac{1}{4}$ in.) is zero. As the plate accelerates downward, this slope must attain a finite value approaching that of the theory, and in doing so the curves are displaced to the right. This accounts for the observation that, for the most part, the experimental curves lie consistently above the corresponding theoretical plots.

The fact that the discrepancy between theory and experiment is largest at the point of maximum curvature of the h vs t curves may be attributed mainly to oil inertia effects. During sinkage, the oil escaping from under the plate attains appreciable velocities and when this flows inward in a convergent manner over the back of the plate, it neutralizes to some extent the hydrostatic pressure causing an effective decrease in applied load which may be as great as 12 %. For parallel sinkage, oil inertia effects satisfactorily explain the larger discrepancy between theory and experiment at larger rather than smaller loads, due to the increased velocity of inflow over the back of the plate.

Because of plate cushioning in the region of maximum curvature of the curves, a downward plate inertia force adds to the nominal plate load opposing the effect of oil inertia. For parallel sinkage, the rate of sinkage is such that the downward plate inertia force is small compared with the effective decrease in applied load due to oil inertia. For inclined sinkage, however, the greater value of dh/dt at the higher load produces a greatly increased downward plate inertia force with little change in the effect due to oil inertia compared with the case of parallel sinkage using the same nominal load. The experimental and theoretical curves in figure 7, therefore, show a lesser discrepancy at the higher rather than at the lower load.

Finally, the order of agreement between theory and experiment is the same for the case of parallel sinkage with a load of 7 lb. and for inclined sinkage with a load of 2 lb. A glance at equations (1) and (9′) shows that the sinkage rate values are about the same (since the value of the polynomial is 3·8) and therefore the inertia effects in both cases have approximately the same magnitude.

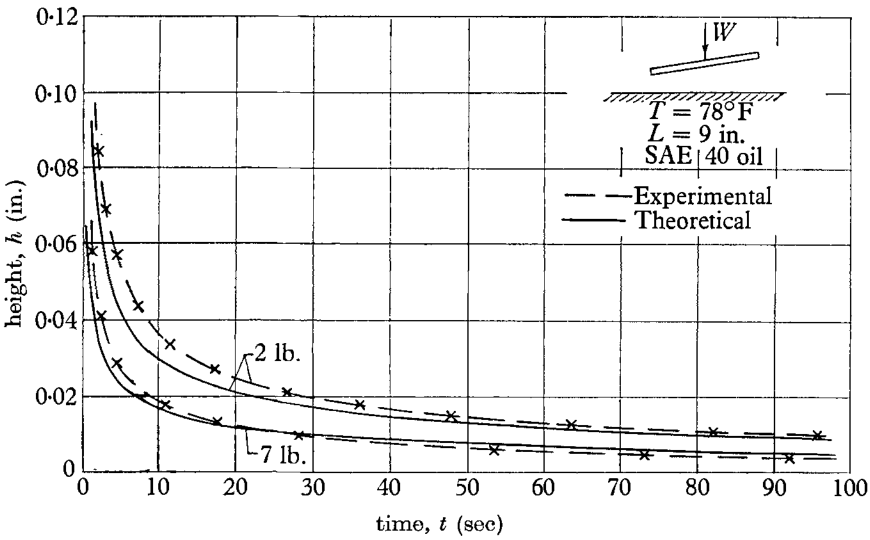

FIGURE 7. Comparison of theoretical and experimental h vs t curves for inclined sinkage.

No experimental data for the case $\beta \neq 0$ have been obtained because of the difficulty experienced with the model of accurately adjusting and maintaining constant during sinkage the selected value of β, when non-zero. This is best illustrated by considering the method used in the tests. Bar stock of appropriate diameter is introduced underneath and along the uppermost edge of the plate, so that in the 'down' or final position the plate is supported by the leading edge and by the bar stock at the opposite edge. The linkage is adjusted to accomplish this and the load acting on the plate is that which will be experienced during the subsequent sinkage tests. If the bar is now removed, it is observed by a movement of the recording gauge that the angle decreases instantaneously. The new value of β can still be measured, of course, by observing the exact number of divisions through which the pointer on the gauge moves. In the actual sinkage tests, however, a further complication is a creeping decrease in β due to flexibility of the linkage. The movement of the gauge pointer thus includes an immeasurable error due to creep in the value of β, and the final resting position of the plate is indeterminate.

When $\beta = 0$, the plate assumes a definite final position, being supported over its entire area rather than along one edge. The existence of creep motion even if a very small value of β is accidentally introduced tends to remove such error, so that it is reasonably certain that indeed $\beta = 0$.

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